# CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
*Evolution Equations in Combinatorics and Physics* :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough, J.-Y. Enjalbert, P. Simonnet.

CIP seminar, Friday conversations:,

For this seminar, please have a look at Slide CCRT[n] & ff.

# Goal of this series of talks.

The goal of these talks is threefold

1. Category theory aimed at “free formulas” and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: colimits
3. Representation theory.
4. MRS factorisation: A local system of coordinates for Hausdorff groups and fine tuning between analysis and algebra.
5. This scope is a continent and a long route, let us, today, walk part of the way together.

# Disclaimers.

**Disclaimer.–** The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

**Disclaimer II.–** The reader will find repetitions and reprises from the preceding CCRT[n], they correspond to some points which were skipped or uncompletely treated during preceding seminars.

# Bits and pieces of representation theory

and how bialgebras arise

## Wikipedia says

Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces .../...
The success of representation theory has led to numerous generalizations. One of the most general is in category theory.

As our track is based on Combinatorial Physics and Experimental/Computational Mathematics, we will have a practical approach of the three main points of view

- Algebraic
- Geometric
- Combinatorial
- Categorical

# Matters

1. Representation theory (or theories)
   1. Geometric point of view
   2. Combinatorial point of view (Ram and Barcelo manifesto)
   3. Categorical point of view
2. From groups to algebras
   *Here is a bit of rep. theory of the symmetric group, deformations, idempotents*
3. Irreducible and indecomposable modules
4. Characters, central functions and shifts.
   *Here are (some of) Lascoux and Schützenberger's results*
5. Reductibility and invariant inner products
   *Here stands Joseph's result*
6. Commutative characters
   *Here are time-ordered exponentials, iterated integrals, evolution equations and Minh's results*
7. Lie groups Cartan theorem
   *Here is BTT*

# CCRT[26] Representation theory and tensor products (deformations and coherence).

## Plan (this and next talks)

1. What are universal problems ? (through examples: w.r.t. a functor, a diagram, Bourbaki's $\alpha$-applications)
2. What is a tensor product ?
3. Free (noncommutative) algebra
4. Fonctors $M \mapsto \mathcal{L}ie_{\mathbf{k}}(M),\ T, S$
5. Another tensor product (Hecke algebra at $q = 0$)
6. Monoidal categories
7. Mc Lane coherence theorem and associators
8. Concluding remarks

# Back to the beginning: Universal Problems, heteromorphisms and adjunctions

1. **With respect to a functor.–**
2. Let $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ be two categories and $F: \mathcal{C}_{right} \to \mathcal{C}_{left}$ a (covariant) functor between them

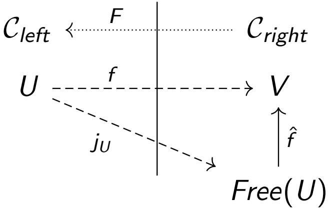



Figure: A solution of the universal problem w.r.t. the functor $F$ is the datum, for each $U \in \mathcal{C}_{left}$, of a pair $(j_U, Free(U))$ (with $j_U \in Hom(U, F[Free(U)])$, $Free(U) \in \mathcal{C}_{right}$) such that, for all $f \in Hom(U, F[V])$ it exists a unique $\hat{f} \in Hom(Free(U), V)$ with $F[\hat{f}] \circ j_U = f$. Elements in $Hom(U, F[V])$ are called heteromorphisms their set is noted $Het_F(U, V)$.

$$\big(\forall f \in Hom(U, F[V])\big)\big(\exists! \ \hat{f} \in Hom(Free(U), V)\big)\big(F(\hat{f}) \circ j_U = f\big)$$

# In this case, the pair $U \to Free(U)$ is, in fact, a functor.

Which, in turn, will prove to be left-adjoint to $F$

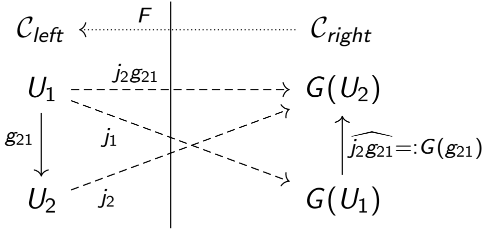



Figure: Making a free functor $G$ $(= Free)$ from $F$: for any morphism $g_{21} \in Hom(U_1, U_2)$, $G(g_{21})$ is the unique morphism in $Hom(U_1, U_2)$ such that

$$F[G(g_{21})] \circ j_1 = j_2 g_{21} \qquad (**)$$

We now prove that $G$ is a functor.

- If $U_1 = U_2$ and $g_{21} = Id_{U_1}$, then $j_1 = j_2 = j_2 g_{21}$ and $F[Id_{G(U_1)}] \circ j_1 = j_1 = j_2 g_{21}$ hence $G[Id_{U_1}] = Id_{G(U_1)}$
- **A remark**: $Het(?, ?)$ is intended to give a symmetric middle term/step to the adjunction chain $Hom(U, F[V]) =: Het_F(U, V) \simeq Het^G(U, V) := Hom(G(U), V)$ $\simeq$ being constructed by a set of bijections.

# Functor $G$ from $Free/2$

- Let now $U_1 \xhookrightarrow{g_{21}} U_2 \xhookrightarrow{g_{32}} U_3$ be a chain of $C_{left}$-morphisms. We have

$$F[G(g_{21})] \circ j_1 = j_2 \circ g_{21} \text{ and } F[G(g_{32})] \circ j_2 = j_3 \circ g_{32}$$

then

$$F[G(g_{32}) \circ G(g_{21})] \circ j_1 \overset{(1)}{=} F[G(g_{32})] \circ F[G(g_{21})] \circ j_1 \overset{(2)}{=} F[G(g_{32})] \circ j_2 \circ g_{21} \overset{(3)}{=} j_3 \circ g_{32} \circ g_{21}$$

(1) because $F$ is a functor, (2) is Eq. (**) applied to indices 21, (3) is Eq. (**) applied to indices 32.
Now, we know that $g \in Hom(U, U')$ being given, the solution $X \in Hom(G(U), G(U'))$ of

$$F[X] \circ j_1 = j_2 \circ g$$

is unique. Then $G(g_{32}) \circ G(g_{21}) = G(g_{32} \circ g_{21})$ □

# Transitivity of free functors.

Piling free structures.

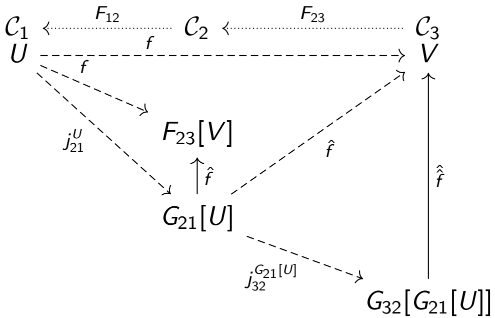
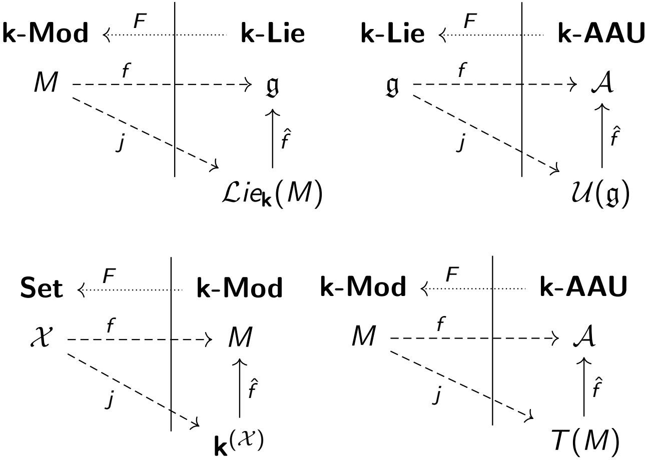



Figure: $\left[F_{12}[j_{32}^{G_{21}[U]}], G_{32}[G_{21}[U]]\right]$ is a solution of the universal problem for $F_{12}F_{23}$ .

Proof: In fact, $Het_{F_{12}F_{23}}(U, V) = Hom(U, F_{12}F_{23}[V]) = Het_{F_{12}}(U, F_{23}[V])$, hence existence of $\hat{f} \in Hom(G_{21}[U], F_{23}[V]) = Het_{F_{23}}(G_{21}[U], V)$, hence again existence of $\hat{\hat{f}}$. Uniqueness of $\hat{\hat{f}}$ is left to the reader.

# First example: $T = UL$.




$T(M) = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}(M))$ $\qquad$ $\mathbf{k}\langle \mathcal{X} \rangle = T(\mathbf{k}^{(\mathcal{X})})$ $\qquad$ $[a, b] := ab - ba$

# An immediate (and although rich) example/1

## Piling free structures/2

1. First, $\mathcal{C}_1 = \textbf{Set}$ (sets and maps) and $\mathcal{C}_2 = \textbf{Mon}$ (monoids and morphisms) gives you the triple $(\mathcal{X}, j_{21}, \mathcal{X}^*)$

Usually $\mathcal{X}$, a set, is seen as an *alphabet* that is to say a *set of non commuting variables.* Let us introduce the ring $\mathbf{k}$ of coefficients

2. With $\mathcal{C}_2 = \textbf{Mon}$ (monoids and morphisms) and $\mathcal{C}_3 = \mathbf{k} - \textbf{AAU}$ ($\mathbf{k}$-associative algebras with unit), one gets $\mathbf{k}[M]$ the algebra of a monoid $M$, we get the triple $(M, j_{32}, \mathbf{k}[M])$ and,
3. by transitivity of free objects with $\mathcal{C}_1 = \textbf{Set}$ (sets and maps) and $\mathcal{C}_3$ as above, we get the triple $(\mathcal{X}, j_{31}, \mathbf{k}\langle\mathcal{X}\rangle)$, $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$ being the algebra of noncommutative polynomials.
4. we immediately obtain that $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$ is free with $\{w\}_{w\in\mathcal{X}^*}$ (this will be useful for the principal pairing)

# An immediate (and although rich) example/2

5. let us observe here that $\mathbf{k}\langle\mathcal{X}\rangle$ can be reached, instead of

$$[\mathbf{Set}] \longrightarrow [\mathbf{Mon}] \longrightarrow [\mathbf{k} - \mathbf{AAU}]$$

   by another path, and this will provide a host of other very interesting (combinatorial) bases.

6. the preceding route amounts to the formula $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$, but it can be also proved that $\mathbf{k}\langle\mathcal{X}\rangle = \mathcal{U}(\mathcal{L}ie\mathbf{k}[X])$

$$[\mathbf{Set}] \longrightarrow [\mathbf{k} - \mathbf{Lie}] \longrightarrow [\mathbf{k} - \mathbf{AAU}]$$

# An immediate (and although rich) example/3

Piling free structures and dual bases

7. from the first (obvious) way (sets to monoids to **k**-AAU) we got the basis $\{w\}_{w\in\mathcal{X}^*}$ which provides the fine grading of $\mathbf{k}\langle\mathcal{X}\rangle$. indeed to each word $w\in\mathcal{X}^*$, we can associate the family

$$\beta(w) = (|w|_x)_{x\in\mathcal{X}} \in \mathbb{N}^{(\mathcal{X})}$$

8. therefore, due to this partitioning of the basis (of words), we get

$$\mathbf{k}\langle\mathcal{X}\rangle = \bigoplus_{\alpha\in\mathbb{N}^{(\mathcal{X})}} \mathbf{k}_\alpha\langle\mathcal{X}\rangle \qquad (1)$$

where $\mathbf{k}_\alpha\langle\mathcal{X}\rangle := span_{\mathbf{k}}\{w|\beta(w)=\alpha\}$.

# An immediate (and although rich) example/4

## Graded bases through free Lie algebra

9. each $\mathbf{k}_\alpha\langle\mathcal{X}\rangle$ is free of dimension $\dfrac{|\alpha|\,!}{\alpha\,!}$; for example with two letters $a, b$, we have $\mathbf{k}\langle\mathcal{X}\rangle = \bigoplus_{(p,q)\in\mathbb{N}^2}\mathbf{k}_{(p,q)}\langle\mathcal{X}\rangle$ and $\dim(\mathbf{k}_{(p,q)}\langle\mathcal{X}\rangle) = \dfrac{(p+q)\,!}{p\,!q\,!} = \dbinom{p}{q}$.
10. this fine grading is a grading of algebra as

$$\mathbf{k}_\alpha\langle\mathcal{X}\rangle\mathbf{k}_\beta\langle\mathcal{X}\rangle \subset \mathbf{k}_{\alpha+\beta}\langle\mathcal{X}\rangle \;;\; 1_{\mathcal{X}^*} \in \mathbf{k}_0\langle\mathcal{X}\rangle \tag{2}$$

11. now through the second route (sets-Lie-AAU), we can construct many finely homogeneous bases of $\mathbf{k}\langle\mathcal{X}\rangle$ using the following scheme
    - pick any finely homogeneous basis of $\mathcal{L}ie\mathbf{k}[X]$, $(P_i)_{i\in I}$ (we will construct at least one)
    - (Totally) order $I$ and form the PBW basis (of $\mathbf{k}\langle\mathcal{X}\rangle$). it is finely homogeneous (due to eq. 2).
    - Use this for MRS factorisation (unfolded below)

# Universal problem without functor: Coproducts

All here is stated within the same category $\mathcal{C}$.

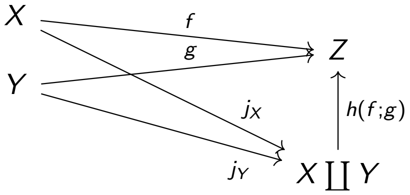



Figure: Coproduct $(j_X, j_Y; X \coprod Y)$.

$$\left(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\right)$$
$$\left(\exists!\, h(f;g) \in Hom(X \coprod Y, Z)\right)$$
$$\left(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\right) \tag{3}$$

# Coproducts: Sets

All here is stated within the category **Set**.

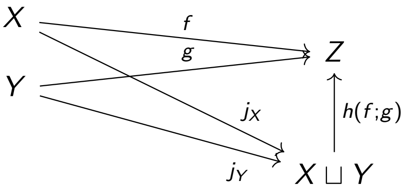



Figure: Coproduct $(j_X, j_Y; X \sqcup Y)$.

$$\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)$$
$$\big(\exists!\, h(f;g) \in Hom(X \sqcup Y, Z)\big)$$
$$\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big) \tag{4}$$

# Coproducts: Modules

All here is stated within the same category **k-Mod**.

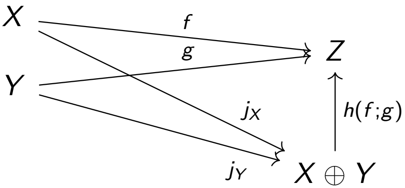



Figure: Coproduct $(j_X, j_Y; X \oplus Y)$ here $h(f;g) = f \oplus g$.

$$\begin{aligned}
&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big) \\
&\big(\exists!\, h(f;g) \in Hom(X \oplus Y, Z)\big) \\
&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)
\end{aligned} \tag{5}$$

# Coproducts: **k-CAAU**

All here is stated within the same category **k-CAAU**.

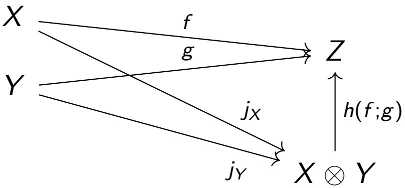



Figure: Coproduct $(j_X, j_Y; X \otimes Y)$ here $h(f;g) = f \otimes g$.

$$\begin{array}{l}(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)) \\ (\exists!\ h(f;g) \in Hom(X \otimes Y, Z)) \\ (h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g)\end{array} \tag{6}$$

# Coproducts: Augmented **k-AAU**

All here is stated within the same category *Augmented* **k-AAU**.

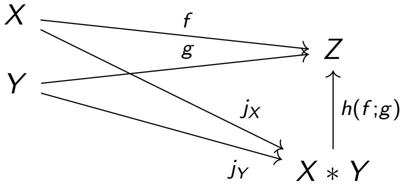



Figure: Coproduct $(j_X, j_Y; X * Y)$ here $h(f;g) = f * g$.

$$\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)$$
$$\big(\exists!\, h(f;g) \in Hom(X * Y, Z)\big)$$
$$\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big) \tag{7}$$

# $\alpha$-applications: tensor products.

9. Here $\mathcal{C}_{left} = \mathbf{k}\text{-}\mathbf{Mod} \times \mathbf{k}\text{-}\mathbf{Mod}$, $\mathcal{C}_{right} = \mathbf{k}\text{-}\mathbf{Mod}$.

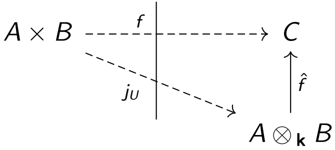



Figure: A solution of the universal problem of tensor products: $A, B, C$ are **k**-modules, $f$ is **k**-bilinear (**k** is a commutative ring and $\hat{f}$ is unique )

10. If you look at the axioms of $\alpha$-applications [2] Ch IV §3.1 (universal sets and mappings). You see that the $\alpha$-applications are kind of left module w.r.t. the endomorphisms of $\mathcal{C}_{right}$ ($\mathrm{QM}_{\mathbf{II}}$ p 283), this left ideal is principal ($\mathrm{AU}'_{\mathbf{I}}$ p 284) and there is unicity of the factorisation ($\mathrm{AU}''_{\mathbf{I}}$ p 284).

11. As regards the case of tensor products, the class of $\alpha$-applications is that of **k**-bilinear mappings from $A \times B \to C$.

# Commutative diagram in a category.

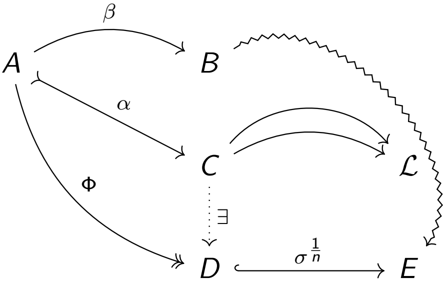
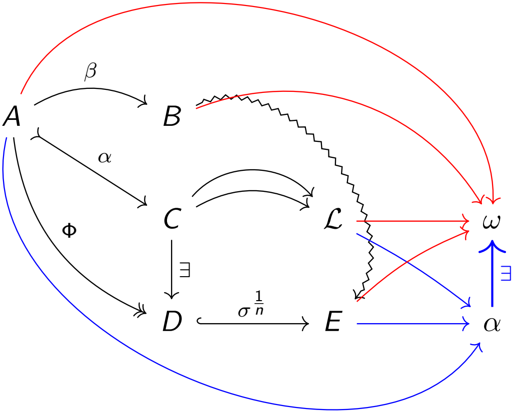



Figure: A commutative diagram is a finite or infinite set of arrows $\mathcal{D}$ (with two maps $\texttt{tail}(?), \texttt{head}(?)$). A path in a diagram is a sequence $a_1, \cdots, a_n$ of arrows of $\mathcal{D}$ such that, for all $1 \leq j < n$, $head(a_j) = tail(a_{j+1})$. The **evaluation** of a path is the composition of its labels. A diagram is said **commutative** iff these evalutations depend only on the endpoints.

# Universal problem without functor: colimit of a commutative diagram.

Covers: disjoint unions, direct sums, coproducts, pushouts and direct limits (inductive limits).



## Algebraic structure

- Concatenation: This law is noted *conc*
- With the empty word as neutral, the set of words is the free monoid $(X^*, conc, 1_{X^*})$
- The pairing between series and polynomials is defined by

$$\langle S|P\rangle = \sum_{w\in\mathcal{X}^*} \langle S|w\rangle\langle P|w\rangle$$

Coding by words gives access to a welter of structures, studies, relations and results (algebra, geometry, topology, probability, combinatorics on words, on polynomials and series). We will use in particular their complete factorisation by Lyndon words.

# Optional: an immediate (and although rich) example/6



Example with $\mathcal{X} = \{a, b\}$, $a < b$, in red Lyndon words ($= \mathcal{L}yn\mathcal{X}$).

| Length | words |
|---|---|
| 0 | $1_{X^*}$ |
| 1 | $a$, $b$ |
| 2 | $aa$, $ab$, $ba$, $bb$ |
| 3 | $aaa$, $aab$, $aba$, $abb$, $baa$, $bab$, $bba$, $bbb$ |
| 4 | $a^4$, $a^3b$, $a^2ba$, $a^2b^2$, $aba^2$, $abab$, $ab^2a$, $ab^3$ |
| | $ba^3$, $ba^2b$, $baba$, $babb$, $b^2a^2$, $b^2ab$, $b^3a$, $b^4$ |

## Two properties of Lyndon words

1. All $\ell \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X}$ factorises (not uniquely in general) as $\ell = \ell_1\ell_2,\ \ell_1 \prec \ell_2,\ \ell_i \in \mathcal{L}yn\mathcal{X}$ (ex. $a^3ba^2bab = a^3b|a^2bab = a^3ba^2b|ab$), the one with the longest right factor will be called standard $\sigma(\ell) = (\ell_1, \ell_2)$.
2. Every word $w \in \mathcal{X}^*$ factorises uniquely $w = \ell_1^{i_1} \ldots \ell_k^{i_k}$ with $\ell_1 \succ \ldots \succ \ell_k, (\ell_i \in \mathcal{L}ynX)$

# Optional: an immediate (and although rich) example/7

Shuffle product(s)

## Non deformed case

Coming from the route where $\mathbf{k}\langle\mathcal{X}\rangle = \mathcal{U}(\mathcal{L}ie\mathbf{k}[X])$, we have a structure of coalgebra on $\mathbf{k}\langle\mathcal{X}\rangle$ its comultiplication is given by its value on letters

$$\Delta_{\sqcup\!\sqcup}(x) = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x \tag{8}$$

Then shuffle product is defined as a dual law, for each $w \in \mathcal{X}^*$ by

$$\langle P \sqcup\!\sqcup Q | w \rangle = \langle P \otimes Q | \Delta_{\sqcup\!\sqcup}(w) \rangle \tag{9}$$

We get the following recursion for shuffle products

$$w \sqcup\!\sqcup 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \sqcup\!\sqcup w = w \qquad \text{for any word } w \in \mathcal{X}^*; \tag{10}$$

$$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) \tag{11}$$

## Lyndon basis

$$\begin{array}{rcl} P_x = & x & \text{for } x \in X, \\ P_\ell = & [P_s, P_r] & \text{for } \ell \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X} \text{ and } \sigma(\ell) = (s, r), \\ P_w = & P_{\ell_1}^{i_1} \ldots P_{\ell_k}^{i_k} & \text{for } w = \ell_1^{i_1} \ldots \ell_k^{i_k}, \ell_1 \succ \ldots \succ \ell_k,\ (\ell_i \in \mathcal{L}yn\mathcal{X}). \end{array}$$

where $\succ$ stands for the lexicographic (strict) ordering defined from $x_0 \prec x_1$.

## Triangular property

Indeed $\{P_w\}_{w \in X^*}$ is lower unitriangular w.r.t. words (this property, joined with the fact that this family is finely homogeneous, implies that $\{P_w\}_{w \in X^*}$ is a basis of $\mathbf{k}\langle\mathcal{X}\rangle$)

$$P_w = w + \sum_{v \succ w, \beta(v) = \beta(w)} c_v v \text{ with } c_v \in \mathbb{Z} \tag{12}$$

# Categorification and decategorification

9. Categorification is better understood through decategorification. Categorification is, so to speak a section of decategorification i.e.

$$[\textbf{Classes}] \xrightarrow{\textit{Categorification}} [\textbf{Category}] \xrightarrow{\textit{Decategorification}} [\textbf{Classes}]$$

10. Decategorification is roughly "taking the isomorphisms classes"
11. A very first example is with $\mathcal{C} = \textbf{FinSet}$ the category of finite sets with $Hom(X, Y) = Y^X$. Here, the isomorphisms classes are indexed by elements of $\mathbb{N}$ (i.e. the cardinality).

# Categorification and decategorification/2

12. But $\mathbb{N}$ is not just a set indexing the isoclasses, it is a semiring $(\mathbb{N}, +, 0, \bullet, 1)$. Decategorification can be performed as follows

| Numbers | *FinSet* |
|---|---|
| $0_{\mathbb{N}}$ | $\emptyset$ |
| $1_{\mathbb{N}}$ | Singletons |
| $+$ | $\sqcup$ (disjoint union) |
| $\bullet$ | $\times$ (cartesian product) |

13. As we have several representatives for each class so we cannot expect true equalities for categorifications of $(x + y).z = x.z + y.z$ i.e.

$$(X \sqcup Y) \times Z \simeq (X \times Z) \sqcup (Y \times Z)$$

# Conclusion

1. We have seen universal constructions w.r.t. a functor and this leads a a left-adjoint of this functor.
2. Some universal constructions are not done w.r.t. a functor but, always, the pattern of unique factorization is kept.
3. In particular coproducts and $\alpha$-applications give rise to such constructions
4. In the next talk we will combine what we have seen with the free magma and tensor products to explain the meaning Mc Lane coherence theorem.

Thank you for your attention.

[1] N. Bourbaki, *Algebra, Chapter 8*, Springer, 2012.

[2] N. Bourbaki, *Theory of Sets*, Springer, 2004.

[3] P. Cartier, *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, Volume 30 (1987-1988) , Talk no. 687 , p. 31-52

[4] What precisely is “categorification“ `https://mathoverflow.net/questions/4841`

[5] M. Deneufchâtel, GD, V. Hoang Ngoc Minh and A. I. Solomon, *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

[6] Jean Dieudonné, *Foundations of Modern Analysis*, Volume 2, Academic Press; 2nd rev edition (January 1, 1969)

[7] GD, Quoc Huan Ngô and Vincel Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

[8] GD, Darij Grinberg, Vincel Hoang Ngoc Minh, *Three variations on the linear independence of grouplikes in a coalgebra*, ArXiv:2009.10970 [math.QA] (Wed, 23 Sep 2020)

[9] Gérard H. E. Duchamp, Christophe Tollu, Karol A. Penson and Gleb A. Koshevoy, *Deformations of Algebras: Twisting and Perturbations* , Séminaire Lotharingien de Combinatoire, B62e (2010)

[10] GD, Nguyen Hoang-Nghia, Thomas Krajewski, Adrian Tanasa, *Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach*, Advances in Applied Mathematics 51 (2013) 345–358.

[11] K.T. Chen, R.H. Fox, R.C. Lyndon, *Free differential calculus, IV. The quotient groups of the lower central series*, Ann. of Math. , 68 (1958) pp. 81–95

[12] V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

[13] M.E. Hoffman, *Quasi-shuffle algebras and applications*, arXiv preprint arXiv:1805.12464, 2018

[14] H.J. Susmann, *A product expansion for Chen Series*, in Theory and Applications of Nonlinear Control Systems, C.I. Byrns and Lindquist (eds). 323-335, 1986

[15] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math, 163, Springer-Verlag (1970).

[16] M. Lothaire, *Combinatorics on Words*, 2nd Edition, Cambridge Mathematical Library (1997).

[17] Szymon Charzynski and Marek Kus, *Wei-Norman equations for a unitary evolution*, Classical Analysis and ODEs, J. Phys. A: Math. Theor. 46 265208

[18] Rimhac Ree, *Lie Elements and an Algebra Associated With Shuffles*, Annals of Mathematics Second Series, Vol. 68, No. 2 (Sep., 1958)

[19] G. Dattoli, P. Di Lazzaro, and A. Torre, $SU(1,1), SU(2)$, *and* $SU(3)$ *coherence-preserving Hamiltonians and time-ordering techniques. Phys.* Rev. A, 35:1582–1589, 1987.

[20] J. Voight, *Quaternion algebras*, `https://math.dartmouth.edu/~jvoight/quat-book.pdf`

[21] Where does the definition of tower of algebras come from ? `https://mathoverflow.net/questions/75787`

[22] Extending Arithmetic Functions to Groups `https://mathoverflow.net/questions/54863`

[23] Where does the definition of tower of algebras come from ?
https://mathoverflow.net/questions/75787

[24] Adjuncts in nlab.
https://ncatlab.org/nlab/show/adjunct